The Levi form and pcvxity

Let us now assume that $M = \partial\Omega \subseteq \mathbb{C}^n$ is a C^2-smooth $(2n-1)$-dim $(\mathbb{R})$ mfld.

Let $\rho = 0$ be a <u>defining function</u> for M, i.e. $\rho : \mathbb{C}^n \to \mathbb{R}$ is C^2, $d\rho \neq 0$ on M, and $M = \{z : \rho(z, \bar{z}) = 0\}$

<u>Note</u>. Any two defining functions $\rho, \tilde{\rho}$ are related by $\rho = a\tilde{\rho}$, where $a \in C^2$ and $a \neq 0$ on M.

convenient notion to remind you that when taking derivatives you need <u>both</u> $\frac{\partial \rho}{\partial z_k}, \frac{\partial \rho}{\partial \bar{z}_\ell}$, etc., to capture all real derivatives $\frac{\partial \rho}{\partial x_k}, \frac{\partial \rho}{\partial y_\ell}$.

<u>Def</u>. For $p \in M$, $\underline{\underline{T_p^{1,0} M}} \subseteq T_p^{1,0}\mathbb{C}^n$ consists of $\xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j} \in T_p^{1,0}\mathbb{C}^n$ s.t.

$$\sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(p)\xi_j = 0.$$

It follows easily from the <u>Note</u> above that T_pM is independent of choice of ρ. Moreover, since $\partial\rho = \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j} dz_j$ does not vanish on M, T_pM is a $(n-1)$-$\dim_{\mathbb{C}}$ subspace of $T_p\mathbb{C}^n \cong \mathbb{C}^n$.

<u>Thm 1</u> Let $\Omega \subseteq \mathbb{C}^n$ and assume $M = \partial\Omega$ is $\mathcal{C}^2$ w/ a defining function ρ s.t. $\Omega = \{\rho < 0\}$. Then, Ω is ψcvx $\Leftrightarrow$ $\forall p \in \partial\Omega$

$$(*) \quad \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}(p)\, \xi_i \bar{\xi}_j \geq 0, \ \forall \xi \in T_p^{1,0}M$$

<u>Pf.</u> We note that $(*)$ is independent of the choice of defining function. If $\tilde{\rho} = a\rho$ is another (with $a > 0$), then

$$\frac{\partial}{\partial z_i}\tilde{\rho} = a\frac{\partial\rho}{\partial z_i} + \rho\frac{\partial a}{\partial z_i}, \quad \frac{\partial^2\tilde{\rho}}{\partial z_i \partial \bar{z}_j} = a\frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}$$

$$+ \frac{\partial\rho}{\partial z_i}\frac{\partial a}{\partial \bar{z}_j} + \frac{\partial\rho}{\partial \bar{z}_j}\frac{\partial a}{\partial z_i} + \rho\frac{\partial^2 a}{\partial z_i \partial \bar{z}_j}.$$

Thus, if $p \in M$ ($\Rightarrow \rho(p) = 0$) and $\xi \in T_pM$ ($\Rightarrow \sum_{i=1}^{n}\frac{\partial\rho}{\partial z_i}\xi_i = \sum_{j=1}^{n}\frac{\partial\rho}{\partial \bar{z}_j}\bar{\xi}_j = 0$), we find

$$\sum_{i,j} \frac{\partial^2 \tilde{\rho}}{\partial z_i \partial \bar{z}_j}(p)\xi_i \bar{\xi}_j = a \sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p)\xi_i \bar{\xi}_j .$$

<u>Lemma 1</u>. If $M = \partial\Omega$ is C^2, then $\delta(z) := d(z, \Omega^c)$ is C^2 up to M for $z \in \Omega$ suff. close to M.

<u>Pf</u>. <u>Ex</u>. (Use Implicit Function Thm). ☒

Thus, we may use a ρ s.t. $\rho = -\delta$ in $\bar{\Omega}$, since by Thm 3 (ii), Ψcvxity is characterized by $-\log\delta$ being PSH near M.

We now proceed w/ pf using $\rho = -\delta$ in $\bar{\Omega}$.

$\Rightarrow$. This is easy. Computing $\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\delta \Rightarrow$

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\delta = \frac{1}{\delta}\frac{\partial^2 \delta}{\partial z_i \partial \bar{z}_j} - \frac{1}{\delta^2}\frac{\partial\delta}{\partial z_i}\frac{\partial\delta}{\partial \bar{z}_j}. \quad (1)$$

If we pick $\xi \in T_pM$, for some $p \in M$, we can extend it to a vector field $\xi(z)$ for $z \in \bar{\Omega}$ near p s.t. $\sum_{i=1}^{n} \frac{\partial\delta}{\partial z_i}(z)\xi_i(z) = 0$

(Since $d\delta \neq 0$ near M, ↑ defines a $(n-1)$-dim$_{\mathbb{C}}$ subbundle of $T\mathbb{C}^n$ for $z \in \bar{\Omega}$ near M)

Since $-\log\delta$ is PSH, for such $\xi(z)$, (1) $\Rightarrow$

$$0 \geq \sum_{i,j} \frac{\partial^2 \delta}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j \Big|_z \quad \text{for } z \in \bar{\Omega} \text{ near } p.$$

By continuity this holds at $p \in M \Rightarrow (*)$ for $\rho = -\delta$.

$\Leftarrow$. We establish this by contradiction. Suppose $-\log\delta$ is not PSH near $\partial\Omega$. Then, we can find $z_0 \in \Omega$, arbitrarily close to M, and $\xi \in \mathbb{C}^n$ s.t. $-\log\delta(z_0 + w\xi)$ is not SH. Setting $v(w) = \log\delta(z_0 + w\xi)$

$\Rightarrow \frac{\partial^2 v}{\partial w \partial \bar{w}}(0) = C > 0$. By Taylor expanding v at 0, (using $v_w = \frac{\partial v}{\partial w}$, etc.)

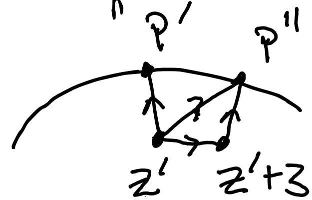
$$v(w) = v(0) + v_w(0) w + \overline{v_w(0) w} + \tfrac{1}{2}\left(v_{ww}(0) w^2 + \overline{v_{ww}(0) w^2}\right) + c|w|^2 + o(|w|^2) = \log\delta(z_0) + \mathrm{Re}(Aw + Bw^2) + c|w|^2 + o(|w|^2)$$

$$\geq \log\delta(z_0) + \mathrm{Re}(Aw + Bw^2) + \varepsilon|w|^2,$$

for suff. small $|w|$ and $\varepsilon > 0$. $\Rightarrow$

$$\delta(z_0 + w\xi) \geq \delta(z_0) \left| e^{Aw + Bw^2} \right| e^{\varepsilon|w|^2} \qquad (2)$$

Next, let $p \in M$ be s.t. $|p - z_0| = \delta(z_0)$ (this can be done suff. close to M by the pf of Lemma 1). Δ-ineq. $\Rightarrow$



$\delta(z') \le |\zeta| + \delta(z'+\zeta)$. Take $z' = z_0 + w\xi$ and $\zeta = (p - z_0)e^{Aw+Bw^2}$ and

$$z(w) = z' + \zeta = z_0 + w\xi + (p - z_0)\, e^{Aw+Bw^2}$$

$$\Rightarrow \quad \delta(z(w)) \ge \delta(z_0 + w\xi) - \delta(z_0)\left|e^{Aw+Bw^2}\right|$$

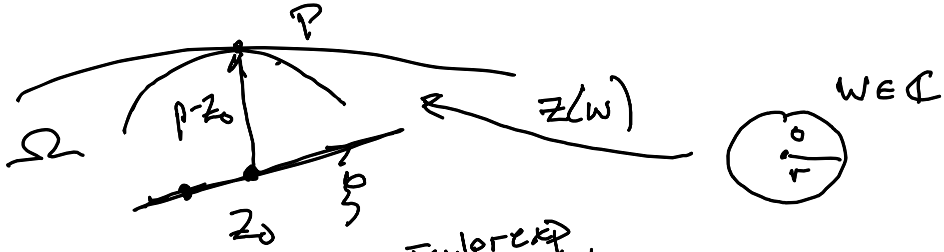
$$\ge \delta(z_0)\, e^{\mathrm{Re}(Aw+Bw^2)}\left(e^{\varepsilon|w|^2} - 1\right)$$

$$(3) \qquad \overset{(2)}{\Uparrow}$$

$$\ge \varepsilon|w|^2 \delta(z_0)\, e^{\mathrm{Re}(Aw+Bw^2)}$$

$$\ge \varepsilon'|w|^2 \quad \text{when } |w| \ll 1.$$

Thus, the <u>holom. disk</u> $w \to z(w)$ is contained in Ω except for $z(0) = p \in M$ (i.e. $w \to z(w)$ is a holom. mapping from a disk $|w| \le r$ into $\mathbb{C}^n$.)



by Taylor exp

One "easily" checks that (3) $\Rightarrow$ $\eta = z'(0) \in T_p^{1,0}M$

and $\displaystyle\sum_{i,j} \frac{\partial^2 \delta}{\partial z_i \partial \bar z_j}(p)\, \eta_i \bar\eta_j \geq \varepsilon' > 0$

But $\rho = -\delta$, which then contradicts the assumption in the thm. $\boxtimes$

Rem. Let us explain the Taylor series argument above in more detail. Let $h(w) = \delta(z(w))$. The conclusion $z'(0) \in T_p^{1,0}M \iff \frac{\partial h}{\partial w}(0) = \sum_{j=1}^{n} \frac{\partial \delta}{\partial z_j}(p) z_j'(0) = 0$, and $\frac{\partial^2 h}{\partial w \partial \bar w}(0) = \sum_{i,j} \frac{\partial^2 \delta}{\partial z_i \partial \bar z_j}(p) z_i'(0)\, \overline{z_j'(0)} \geq \varepsilon'$.

But $h(0) = \delta(p) = 0$, so Taylor $\Rightarrow$

$$\operatorname{Re}(h_w(0)w) + \frac{1}{2}\operatorname{Re}(h_{ww}(0)w^2) + h_{w\bar w}(0)|w|^2 \geq \varepsilon' |w|^2 + O(|w|^3) \qquad (4)$$

First, unless $h_w(0) = 0$, the leading term is $\operatorname{Re}(h_w(0)w)$ which will take both positive and negative values near $w=0$, which contradicts (4)! Now, rewrite what is left

$$\mathrm{Re}\, h_{ww}(0) w^2 \geq (\varepsilon' - h_{w\bar{w}}(0)) |w|^2 + O(|w|^3)$$

But $\mathrm{Re}\, h_{ww}(0) w^2$ is harmonic and $=0$ at $w=0$. By Mean Value Thm $\Rightarrow$ $\mathrm{Re}\, h_{ww}(0) w^2$ is either 0 or takes both pos. and neg. terms near $w=0$. In either case, RHS above must allow this $\Rightarrow$ $\varepsilon' - h_{w\bar{w}}(0) \leq 0 \Rightarrow h_{w\bar{w}}(0) \geq \varepsilon'$ as desired.